1. Verify that the standard deviation is square root of variance.

2. Verify that $cov(x, y) = \overline{xy} - \overline{x}\,\overline{y}$ and that correlation coefficient $|C_{xy}| \leq 1$

3. Derive the Maxwell's relations as listed on page 6 of the lecture notes.

2.6   More rigorously, for two variables (x, y)

$$\text{Var}(x+y) = (\langle x-\bar{x}\rangle^2) + (\langle y-\bar{y}\rangle^2) + 2\langle (x-\bar{x})(y-\bar{y})\rangle \geq 0$$

$$= (\langle x-\bar{x}\rangle^2) + (\langle y-\bar{y}\rangle^2) + 2[\overline{xy} - \bar{x}\bar{y}] \geq 0$$

We can then write

$$\text{Var}\left(\frac{x}{\sigma_x} + \frac{y}{\sigma_y}\right) = \frac{\text{Var } x}{\sigma_x^2} + \frac{\text{Var } y}{\sigma_y^2} + \frac{2[\overline{xy} - \bar{x}\bar{y}]}{\sigma_x \sigma_y} \geq 0$$

or $\qquad\qquad 2 + 2[C_{xy}] \geq 0 \quad$ or $\qquad C_{xy} \geq -1$

And for
$$\text{Var}\left(\frac{x}{\sigma_x} - \frac{y}{\sigma_y}\right) = \frac{\text{Var}}{\sigma_x^2} + \frac{\text{Var } \sigma_y}{\sigma_y^2} - \frac{2[\overline{xy} - \bar{x}\bar{y}]}{\sigma_x \sigma_y} \geq 0$$

$$2 - 2[C_{xy}] \geq 0 \quad \text{or} \quad C_{xy} \leq 1$$

$$Q.E.D$$

1. Given variance $V = \langle (x-\mu)^2 \rangle$

expand $V = \langle x^2 + \mu^2 - 2x\mu \rangle = \langle x^2 \rangle + \mu^2 - 2\langle x \rangle \mu$

$$= \langle x^2 \rangle - \mu^2.$$

or $\sqrt{V} = \left[\langle x^2 \rangle - \langle x \rangle^2 \right]^{1/2} = \sigma$

Q.E.D.

2. (a) $Cov(x,y) = \frac{1}{N} \cdot \sum (x_i - \bar{x})(y_i - \bar{y})$

$$= \frac{1}{N} \left[\sum x_i y_i - \sum x_i \bar{y} - \sum \bar{x} y_i + \boxed{\sum \bar{x} \bar{y}} \right]$$

$$= \overline{xy} - \bar{x}\bar{y} - \bar{x}\bar{y} + \bar{x}\bar{y} = \overline{xy} - \bar{x}\bar{y}$$

(b) Easyways: For uncorrelated $\overline{xy} = \bar{x}\bar{y}$; $C_{xy} = 0$ for uncorrelated.

For maximum correlation (self-correlation) set

$$C(x,x) = \frac{Cov(x,x)}{\sigma_x \sigma_x} = \frac{\overline{x^2} - \bar{x}^2}{\sigma_x^2} = 1$$

3. (i) With S, V as independent variables, we have

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV = TdS - pdV$$

for quasi-static infinitesimal processes.

with 'V' held constant; $T = \left(\frac{\partial U}{\partial S}\right)_V$

with 'S' held constant; $p = -\left(\frac{\partial U}{\partial V}\right)_S$.

Thus $\frac{\partial}{\partial V}\left(\frac{\partial U}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial U}{\partial V}\right)$ leads to

$$\left(\frac{\partial T}{\partial V}\right) = -\frac{\partial P}{\partial S} \qquad \text{Maxwell-1.}$$

(ii) We have Helmholtz's free energy

$$F = U - TS$$

and $dF = dU - TdS - SdT = -pdV - SdT$

or $p = -\left(\frac{\partial F}{\partial V}\right)_T$; $S = -\left(\frac{\partial F}{\partial T}\right)_V$

From the same rules as above,

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T \qquad \text{Maxwell-2}$$

3. (iii)   Gibb's free energy $G = U - TS + PV$

$$dG = dU - TdS - SdT + PdV + VdP$$

$$= -SdT + VdP$$

or $S = -\left(\dfrac{\partial G}{\partial T}\right)_P$; $V = \left(\dfrac{\partial G}{\partial P}\right)_T$

Thus. $\left(\dfrac{\partial S}{\partial P}\right)_T = -\left(\dfrac{\partial V}{\partial T}\right)_P$ Maxwell - 3.

(iv)   Enthalpy $H = U + PV$

$$dH = dU + PdV + VdP = TdS + VdP$$

or $T = \left(\dfrac{\partial H}{\partial S}\right)_P$; $V = \left(\dfrac{\partial H}{\partial P}\right)_T$

and so $\left(\dfrac{\partial T}{\partial P}\right)_S = \left(\dfrac{\partial V}{\partial T}\right)_P$ Maxwell - 4.

1. Find the inversion temperature of the van der Wall's gas.

2. Derive a relation between the thermal expansion coefficient and the compressibility for a van der Wall gas (see page 8 of notes)

3. For an ideal gas, show that the enthalpy at a temperature T is

$$H(T) = H(T_0) + C_P * (T-T_0)$$

1 Find the inversion temperature for Vander Waal's gas

$$\left(P + \frac{a\nu^2}{V^2}\right)(V - \nu b) = \nu RT$$

$$\mu_{JL} = \frac{1}{C_P}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right]$$

(a) inversion temp $\mu_{JK} = 0$

$$\frac{\partial V}{\partial T}\left[\left(P + \frac{a\nu^2}{V^2}\right) - \frac{2a\nu^2}{V^3}(V - \nu b)\right] = \nu R$$

or $$\frac{\partial V}{\partial T}\left[\frac{\nu RT}{(V - \nu b)} - \frac{2a\nu^2}{V^3}(V - \nu b)\right] = \nu R$$

or $$T\frac{\partial V}{\partial T} - V = \frac{\nu RT(V - \nu b)}{\left[\nu RT - 2a\nu^2\frac{(V - \nu b)^2}{V^3}\right]} - V$$

$$= \frac{-\nu RTb + 2a\nu^2 V\frac{(V - \nu b)^2}{V^3}}{[\quad]}$$

For inversion temp, require that the numerator is zero

or $$T = 2a\nu^2\frac{(V - \nu b)^2}{V^2} \cdot \frac{1}{\nu^2 Rb}$$

For most gases $V \gg \nu b$; Then $T_{inv} = \frac{2a}{Rb}$

2. From Van der Waals equ:

$$\left(P + \frac{a\nu^2}{U^2}\right)(V - \nu b) = \nu RT$$

$$\left(\frac{dV}{dT}\right)_P = \frac{\nu R}{\left[\left(P + \frac{a\nu^2}{U^2}\right) - \frac{2a\nu^2}{V^3}(V - \nu b)\right]}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{1}{(V - \nu b)}\left[\frac{2a\nu^2(V - \nu b)}{V^3} - \left(P + \frac{a\nu^2}{U^2}\right)\right]$$

Thus $\alpha/k = \dfrac{\nu R(V - \nu b)}{\left[\quad A \quad\right]^2}$

$A = \left[\left(P + \frac{a\nu^2}{U^2}\right) - \frac{2a\nu^2}{V^3}(V - \nu ..$

Also $\alpha \cdot k = \dfrac{\nu R}{V^2(V - \nu b)}$

a simpler & perhaps more interesting than α/k.

9

We have

$$H = E + PV$$

$$dH = dE + p\,dV + V\,dp = T\,ds + V\,dp$$

With entropy as $S(P, T)$

$$ds = \left(\frac{\partial S}{\partial p}\right)_T dp + \left(\frac{\partial S}{\partial T}\right)_P dT$$

$$\therefore dH = T\left[\left(\frac{\partial S}{\partial p}\right)_T dp + \left(\frac{\partial S}{\partial T}\right)_P dT\right] + V\,dp$$

$$= C_p\,dT + \left[T\left(\frac{\partial S}{\partial p}\right)_T dp + V\,dp\right]$$

From Maxwell ; $\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$

$$\therefore dH = C_p\,dT + \left[V - T\left(\frac{\partial V}{\partial T}\right)_P\right] dp$$

For ideal gas $PV = \nu RT$ or $T\left(\frac{\partial V}{\partial T}\right)_P = \frac{\nu RT}{P} = V$

So second term on R.H.S. vanishes

$$\therefore dH = C_p\,dT \quad \text{for ideal gas}$$

$$\int_{H_0}^{H} dH = C_p \int_{T_0}^{T} dT \qquad \therefore C_p = \text{Constant for ideal gas.}$$

$$H(T) - H(T_0) = C_p(T - T_0) \qquad \text{with } H_0 = H(T_0)$$

Q.E.D.

1. For a system of fixed volume in thermal contact with a reservoir, show that the mean square energy is given by

$$\left\langle \varepsilon_s^2 \right\rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}$$

2. At critical point (T_c), there are unique values for volume (V_c) and pressure (P_c)

At the critical point, the first and second derivatives of pressure with respect to volume vanish

ie. $$\left(\frac{\partial P}{\partial V} \right)_{T_c} = \left(\frac{\partial^2 P}{\partial V^2} \right)_{T_c} = 0$$

For a van der Waal's gas show that

$$P_c = \frac{a}{27b^2} \; ; V_c = 3b \; ; RT_c = \frac{8a}{27b}$$ setting $v=1$

We may deduce the same result by noting $(V-V_c)^3 = 0$. Expand this and the van der Waal's equation in powers of V and require that for $V=V_c$, the three roots of the equations are one and the same.

3. Assume that hydrogen atom has two excited states, the first one at 10 eV and the second one at 12 eV. Calculate the relative population of these two levels with respect to ground states for stellar interiors of temperatures 10^4 K, 10^5 K and 10^6 K. Show that at temperatures above 50 million Kelvin, all three levels (g.s., first and second excited levels) are <u>nearly</u> equally populated.

1.

We have

$$\langle \epsilon_s \rangle = -\frac{1}{Z} \frac{\partial Z}{\partial \beta}$$

And $\langle \epsilon_s^2 \rangle = \sum \frac{\epsilon_s^2 e^{-\beta \epsilon_s}}{Z}$

$$\frac{\partial^r e^{-\beta \epsilon_s}}{\partial \beta^r} = \epsilon_s^2 e^{-\beta \epsilon_s}$$

or $\langle \epsilon_s^r \rangle = \frac{1}{Z} \sum \frac{\partial^r e^{-\beta \epsilon_s}}{\partial \beta^r} = \frac{1}{Z} \frac{\partial^r}{\partial \beta^r} \left(\sum e^{-\beta \epsilon_s} \right)$

$$= \frac{1}{Z} \frac{\partial^r Z}{\partial \beta^r}$$

(2)

Van der Waals' Eqn.

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \qquad \text{setting } v = 1$$

(1 moler content)

Need to evaluate & set $\frac{dP}{dV} = 0$, $\frac{d\tilde{P}}{dv} = 0$ @ $T = T_c$

$$\left[\frac{\partial P}{\partial V} - \frac{2a}{V^3}\right](V - b) + \left(P + \frac{a}{V^2}\right) = 0 \qquad \text{①}$$

@ $T = T_c$

and $\left[\frac{\partial^2 P}{\partial v^2} + \frac{6a}{V^4}\right](V - b) + \left(\frac{\partial P}{\partial V} - \frac{2a}{V^3}\right) + \left(\frac{\partial P}{\partial V} - \frac{2a}{V^3}\right) = 0 \qquad \text{②}$

From ① and ② setting $\partial P/\partial V = 0$ & $\partial^2 P/\partial v^2 = 0$, we have

$$\left(P + \frac{a}{V^2}\right) = \frac{2a}{V^3}(V - b) \qquad \text{③}$$

$$\frac{6a}{V^4}(V - b) = 4\frac{a}{V^3} \qquad \text{④}$$

From ④, we get $V = V_c = 3b$

Substituting this in ③, we get $P_c = \frac{a}{27b^2}$

Substituting V_c & P_c values in E·O·S, we get $T_c = \frac{8a}{27bR}$.

Alternately, we have

$$(V - V_c)^3 = 0 \quad : \quad V^3 - 3V^2 V_c + 3V V_c^2 - V_c^3 \quad \text{⑤}$$

and expanding Vander waals' we have

$$PU^3 - V^2(bP + RT) + aV - ab = 0$$

or

$$V^3 - V^2\left(\frac{bP + RT}{P}\right) + \frac{a}{P}V - \frac{ab}{P} = 0 \qquad \text{⑥}$$

Need that the coefficients of each power of 'V' are same in ⑤ and ⑥

Or

$$V_c^3 = \frac{ab}{P} \qquad \text{⑦}$$

$$3 V_c^2 = \frac{a}{P} \qquad \text{⑧}$$

$$3 V_c = \frac{bP + RT}{P} \qquad \text{⑨}$$

$\left(\frac{Eq7}{Eq8}\right)$ given $\quad V_c = 3b$

From ⑦ $\quad P_c = \frac{ab}{V_c^3} = \frac{a}{27b^2}$

Then from ⑨ we have $\quad RT_c = \frac{8a}{27b}$

(3)

The partition function

$$Z = 1 + e^{-10\beta} + e^{-12\beta}$$

$$\beta = \frac{1}{kT}$$

$$k = 8.62 \times 10^{-5} \, eV/k$$

$$P_{g.s} = \frac{1}{Z}$$

$$P_{E_1}(10eV) = \frac{e^{-10\beta}}{Z}$$

and $P_{E_2}(12eV) = \frac{e^{-12\beta}}{Z}$.

Calculate these probabilities numerically for the temps 1
interest.

T(K)				g.s.	1st exc.	2nd exc.
10000	0.861	9.04E-06	8.85E-07	#REF! 1.00	9.03E-06	8.85E-07
50000	4.305	9.80E-02	6.16E-02	8.62E-01	8.45E-02	5.31E-02
100000	8.61	3.13E-01	2.48E-01	6.41E-01	2.01E-01	1.59E-01
150000	12.915	4.61E-01	3.95E-01	5.39E-01	2.48E-01	2.13E-01
200000	17.22	5.59E-01	4.98E-01	4.86E-01	2.72E-01	2.42E-01
500000	43.05	7.93E-01	7.57E-01	3.92E-01	3.11E-01	2.97E-01
1000000	86.1	8.90E-01	8.70E-01	3.62E-01	3.23E-01	3.15E-01
50000000	4305	9.98E-01	9.97E-01	3.34E-01	3.33E-01	3.33E-01

The last three columns show the population of g.s., 1st excited level and 2nd excited level, respectively. At temps 50 MK and above, all three levels are equally populated.

Show that the dispersion of number of particles in a state "s" is given by

$$\left\langle (\Delta n_s)^2 \right\rangle = \frac{1}{\beta^2 Z} \frac{\partial^2 Z}{\partial \varepsilon_s^2} - \frac{1}{\beta^2} \left(\frac{\partial \ln Z}{\partial \varepsilon_s} \right)^2$$

and it may also be written as

$$\left\langle (\Delta n_s)^2 \right\rangle = -\frac{1}{\beta} \frac{\partial \bar{n}_s}{\partial \varepsilon_s}$$

1.

$$Z = \sum e^{-\beta(n_1 \epsilon_1 + \cdots n_s \epsilon_s \cdots)}$$

$$\overline{n_s} = \frac{\sum n_s e^{-\beta n_s \epsilon_s}}{Z}$$

$$n_s e^{-\beta n_s \epsilon_s} = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_s} e^{-\beta n_s \epsilon_s}$$

or $\overline{n_s} = -\frac{1}{\beta} \frac{1}{Z} \frac{\partial}{\partial \epsilon_s} \sum e^{-\beta n_s \epsilon_s} = -\frac{1}{\beta} \frac{1}{Z} \frac{\partial Z}{\partial \epsilon_s}$

$$= -\frac{1}{\beta} \frac{\partial \log Z}{\partial \epsilon_s}.$$

2. The number dispersion

$$\overline{(n_s - \bar{n}_s)^2} = \langle n_s^2 \rangle - (\bar{n}_s)^2$$

$$\overline{n_s^2} = \frac{\sum n_s^2 \, e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots n_s\epsilon_s + \cdots)}}{Z}$$

Note:
$$\frac{\partial^2}{\partial \epsilon_s^2} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots n_s\epsilon_s)} = \beta^2 n_s^2 \, e^{-\beta(n_1\epsilon_1 + \cdots}$$

$$\therefore \quad \overline{n_s^2} = \frac{1}{\beta^2} \cdot \frac{1}{Z} \frac{\partial^2}{\partial \epsilon_s^2} \left(\sum e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)} \right)$$

$$= \frac{1}{\beta^2} \cdot \frac{1}{Z} \frac{\partial^2 Z}{\partial \epsilon_s^2}$$

or the dispersion $= \dfrac{1}{\beta^2} \dfrac{1}{Z} \dfrac{\partial^2 Z}{\partial \epsilon_s^2} - \dfrac{1}{\beta^2} \left(\dfrac{\partial \ln Z}{\partial \epsilon_s} \right)^2$ ①

① may be further simplified as on the next page

we have $\dfrac{\partial z}{\partial \epsilon_s} = z \dfrac{\partial \ln z}{\partial \epsilon_s}$

∴ $\dfrac{\partial^r z}{\partial \epsilon_s{}^r} = \left(\dfrac{\partial z}{\partial \epsilon_s}\right)\left(\dfrac{\partial \ln z}{\partial \epsilon_s}\right) + z \dfrac{\partial^r \ln z}{\partial \epsilon_s{}^2}$

or $\overline{n_s^2} = \dfrac{1}{\beta^2} \cdot \dfrac{1}{z}\left[\dfrac{\partial z}{\partial \epsilon_s}\left(\dfrac{\partial \ln z}{\partial \epsilon_s}\right) + z \dfrac{\partial^r \ln z}{\partial \epsilon_s{}^r}\right]$

$= \dfrac{1}{\beta^2}\left(\dfrac{\partial \ln z}{\partial \epsilon_s}\right)^2 + \dfrac{1}{\beta^2} \dfrac{\partial^r \ln z}{\partial \epsilon_s{}^2}$

∴ Deviation $\overline{n_s^2} - \langle n_s\rangle^r = \dfrac{1}{\beta^2} \dfrac{\partial^r \ln z}{\partial \epsilon_s{}^2}$.

$= \dfrac{1}{\beta^r} \dfrac{\partial}{\partial \epsilon_s} \cdot \left(-\beta \, \overline{n_s}\right) = -\dfrac{1}{\beta}\dfrac{\partial \overline{n_s}}{\partial \epsilon_s}$.

1. Planck's distribution and that of Wien differ in that Planck has an extra -1 in the denominator. Derive the Wein's displacement law from Wien's distribution. Compare the numerical value of Wien's constant thus deduced with literature value.

2. Derive Stefan-Boltzmann's law from Planck's distribution. Again compare the constant you deduce with the literature value.

3. Solar constant $= 0.136$ J s^{-1} cm^{-2} is the total amount of radiant energy flux density, integrated over all emission wavelengths, at the Earth from the Sun normal to the incident rays, referred to the mean Earth –Sun distance. Given that a) Earth-Sun distance is 1.5×10^{13} cm, b) radius of sun is 7×10^{10} cm and you know Stefan-Boltzmann constant, calculate

   a. The total power generated from the Sun

   b. the effective temperature of the surface of sun (Sun is a blackbody)

   c. The surface temperature of the earth, considering it to be a blackbody in thermal equilibrium. It reradiates as much thermal radiation as it received from the Sun.

1.   Planck's radiation law:

$$U_\nu\, d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu\, d\nu}{\left[e^{h\nu/kT}-1\right]}$$

$$= \frac{8\pi h}{c^3}\frac{\nu^3\, d\nu}{\left[e^{h\nu/kT}-1\right]}$$

To derive Wien's displacement law, ignore '1' in the denominator (good for too high ν and low T) and find max for U_ν.

$$U_\nu = \frac{8\pi h}{c^3}\nu^3 e^{-h\nu/kT}$$

or

$$\frac{dU_\nu}{d\nu} = \frac{8\pi h}{c^3}\left[3\nu^2 + \nu^3\left(-\frac{h}{kT}\right)\right]e^{-h\nu/kT}$$

For max., we have $\dfrac{h\,\nu_{max}}{kT} = 3$

or $T\cdot\lambda_{max} = \dfrac{hc}{3k} = \dfrac{1242\times10^{-9}\times\,ev\text{-}m}{3\times0.861\times10^{-5}\,ev\text{-}k}$?

$$\approx 4.8\times10^{-3}\ m\text{-}k.$$

Stefan- Boltzman law:

2.

$$I = \int u_\nu d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

Put $x = \frac{h\nu}{kT}$

$$I = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$= \frac{8\pi (kT)^4}{(hc)^3} \cdot \frac{\pi^4}{15}$$

$$= \frac{8\pi \cdot \pi^4}{15} \frac{(1.38 \times 10^{-23})^4 T^4}{[1242 \times 1.6 \times 10^{-19} \times 10^{-9}]^3}$$

$$\underline{I} = 7.55 \times 10^{-16} T^4 \text{ Joules.}$$

To compare with Stefan- Boltzman's Law, we have to deduce power. An extra factor of $c/4$ multiplies the above result (See p. 58 of class notes).

Power: $\dfrac{Ic}{4} = 5.65 \times 10^{-8} \text{ W/(m}^2 k^4)$

Compares extremely well with standard value of $\sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} k^{-4}$.

2.   Solar Constant $= 0.136$ $J\ s^{-1}\ cm^{-2}$

                 $=$ Radiant energy flux from Sun to Earth's surface.

(a)

     Total energy generated $=$ Solar Constant $\cdot 4\pi \cdot D^r$

       where $D =$ Sun $-$ earth distance.

     Integrate an ~~area~~ the surface of a sphere of radius D.

     $\therefore$ $\phi = 4\pi\ D^r\ \sigma$ $D = 1.5 \times 10^{13}\ cm$

     $\therefore$ $\phi = 4\pi \times 0.136 \times (1.5 \times 10^{13})^2 = 3.8 \times 10^{26}\ J/s.$

(b)   If $R_\odot$ is the radius 2 Sun, the flux density at the Surface 1 Sun is $J_\odot = \phi / 4\pi R_\odot^2$ $J\ s^{-1}\ cm^{-2}$

     Given $R_\odot = 7 \times 10^{10}\ cm$.

     $J_\odot = \dfrac{3.8 \times 10^{26}}{[4\pi \times 7^2 \times 10^{20}]}$ J/sec^{-1}

        $= 6170\ J\ s^{-1}\ cm^{-2}.$

   a la Stefan Boltzmann; $T = \left[J_\odot / \sigma_B \right]^{\frac{1}{4}} = \left[\dfrac{6170}{5.67 \times 10^{-12}} \right]^{\frac{1}{4}}$

(C) The power absorbed by the earth's surface

$$= \text{solar constant} \times \pi R_{\oplus}^2$$

Power reemitted by earth $= \sigma T^4 \times 4\pi R_{\oplus}^2$

In Equilibrium.

$$\sigma T^4 4\pi R_{\oplus}^2 = \text{Solar const} \times \tau R_{\oplus}^2$$

or $T^4 = \left[\dfrac{0.136 \times 10^4}{5.67 \times 10^{-8} \times 4} \right]$

or $T = 278 K$.

A bit chilly up there!

1. Wien's distribution of blackbody radiation is same as that of Planck without −1 in the denominator. Write it with the wavelength (λ) as the variable, in stead of frequency. Deduce the Wien's displacement law. Calculate the λ_{max} ν_{max} using the result from this calculation and previous calculation. Any surprises? Explain.

2. A process is called Markovian process, if the probability that an event occurs in a time interval dt does not depend on the past history of the system. Say, the probability that a radioactive atom decays in a time interval dt after surving a time t depends only on the magnitude of dt and not on the magnitude of t. Similarly, a photon interacting in a material of thickness dx depends only dx but not on x_0, the distance photon traveled before. Show that these phenomena lead to exponential laws for large time intervals and length intervals etc.

1.

We know

$$u_\nu \, d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} \, d\nu$$

For Wien's we drop -1 in the denominator

$$u_\nu \, d\nu = \frac{8\pi}{c^3} h\nu^3 e^{-h\nu/kT} \, d\nu$$

Convert this to wavelength as variable

$$\lambda = \frac{c}{\nu} \qquad \text{or} \qquad d\lambda = -\frac{c}{\nu^2} d\nu \qquad \left(\begin{array}{l} -\text{ve sign is of} \\ \text{consequence} \end{array} \right)$$

Thus $u_\lambda \, d\lambda = \frac{8\pi h}{c^3} \cdot \frac{c^3}{\lambda^3} \cdot e^{-\frac{hc}{\lambda kT}} \cdot \frac{c}{\lambda^2} \, d\lambda$

or $u_\lambda = \frac{8\pi h c^4}{\lambda^5} e^{-\frac{hc}{\lambda kT}}$

$$\frac{du_\lambda}{d\lambda} = 8\pi hc^4 \left\{ \frac{-5}{\lambda^6} + \frac{hc}{kT} \cdot \frac{1}{\lambda^7} \right\} e^{-hc/\lambda kT} = 0 \quad \text{for } \lambda = \lambda_p \text{ most}$$
$$\text{probable}$$
$$\text{wavelength}$$

or $\lambda_{max} = \frac{hc}{5kT}$

In the previous assignment: $\nu_{max} = \frac{3kT}{h}$

So, we have $\lambda_{max} \, \nu_{max} = \frac{3c}{5}$.

The transformation between λ & ν variables is not linear. We found that Wien's constant as deduced from ν-variable is $\sim \frac{5}{3}$ factor larger than $\lambda_p T = 2.898 \times 10^{?}$ mk. This discrepancy is now clearly due

2. For the Markovian processes, the probability that an atom decays in a time interval dt is $\lambda\, dt$.

or decay Probability in $dt = \lambda\, dt$ $\lambda =$ Proportionality Constant

Survival Probability = 1 − decay Probability.

Probability that

∴ Atom survives a time interval $dt = 1 - \lambda\, dt$

Probab. that it survives the Second time interval $dt =$

Prob. surviving 1st interval × Prob. surviving 2nd interval

$$= (1 - \lambda\, dt)(1 - \lambda\, dt) = (1 - \lambda\, dt)^2.$$

∴ Probability that it survives 'n' consecutive time intervals of 'dt' each such that $t = n\, dt$ is the time of observation

is $$(1 - \lambda\, dt)^n = \left(1 - \frac{\lambda t}{n}\right)^n$$

$$\underset{n\to\infty}{Lt}\left(1 - \frac{\lambda t}{n}\right)^n = e^{-\lambda t} \quad (\text{Def. of exponential function})$$

If we start with N_0 atoms at $t_1 = t_0 = 0$ (arbitrary ref.), the no. of atoms surviving after time $t = t$ is given

$N =$ initial no. of atom × Survival Probability $= N_0 e^{-\lambda t}$

QED

For photon interaction the same argument with time replaced by distance traversed.

1. We may assume that a white dwarf is made up of degenerate electrons and non-degenerate protons. The order of magnitude of the gravitational self-energy is $-GM^2/R$, where G is the gravitational constant, M and R are the mass and radius of the white dwarf, respectively. The mass of the white dwarf is mainly due to protons.

Show that the order of magnitude of the kinetic energy of the electrons in the ground state (see page 70 of the class notes) is

$$\frac{\hbar^2 N^{5/3}}{m_e R^2} \approx \frac{\hbar^2}{m_e R^2}\left(\frac{M}{m_p}\right)^{5/3}$$

where m_e and m_p are the masses of electron and proton, respectively.

Show also that if the gravitational and kinetic energies are of the same order of magnitude,

2. For ultra-relativistic electrons whose rest mass 'm' is much smaller than the total energy, we might write $\varepsilon \approx pc$, where 'p' is the momentum. If we have a number of electrons confined in a cube of volume $V = L^3$, where L is the side of the cube, the momentum is

$$p = \hbar k = \frac{\pi\hbar}{L}\left(n_x^2 + n_y^2 + n_z^2\right)^{1/2}$$,same as in the case of non-relativistic particles

Show that the Fermi energy of relativistic electron gas is given by

$$\varepsilon_F = \hbar\pi c\left(\frac{3n}{\pi}\right)^{1/3}$$

where n= N/V is the electron number density (no. of electrons per unit volume).

Show also that the total energy of the ground state of the gas is

$$U_0 = \frac{3}{4}N\varepsilon_F$$

1. $U_0 = \frac{3}{5} N \epsilon_F = \frac{3}{5} \frac{\hbar^2}{2m} \left(\frac{\pi n_F}{L} \right)$

$\qquad = \frac{3}{5} N \frac{\hbar^2}{2m} \left(\frac{\pi}{L} \right)^2 \cdot \left(\frac{3N}{\pi} \right)^{2/3}$

$\qquad = \frac{\hbar^2}{m} \cdot \frac{1}{R^2} \frac{3 \cdot 3^{2/3} \pi^{4/3}}{10} \cdot N^{5/3}$

$\qquad \qquad \frac{\hbar^2}{m} \frac{N^{5/3}}{R^2} \cdot \left(\frac{3\pi}{10} \right)^{4/3} \simeq \frac{\hbar^2 N^{5/3}}{m R^2}$

If the mass of white dwarf is made up of hydrogen,

then it is only protons that make any significant contribution

to its mass ($\because m_e / m_H = 1/1840$)

$\therefore \quad M = N \cdot M_H$.

$\therefore \quad U_0 \simeq \frac{\hbar^2 N^{5/3}}{m R^2} = \frac{\hbar^2}{m R^2} \left(\frac{M}{M_H} \right)^{5/3}$

If gravitational $PE = 2 \cdot k \cdot E$ ......(the Virial theorem)

we have

$\qquad \frac{3}{5} G \frac{M^2}{R} = U_0 = \frac{\hbar^2 M^{5/3}}{m M_H^{5/3} R^2}$

$\text{or} \qquad M^{1/3} R = \frac{5}{3} \cdot \frac{1}{G} \cdot \frac{\hbar^2}{m M_H^{5/3}}$

$\qquad \quad G = 6.67 \times 10^{-11} \; m^3 kg^{-1} s^{-2}$

$\qquad \quad \hbar = 1.05 \times 10^{-34} \; J sec.$

$\qquad \quad m = 9.1 \times 10^{-31} \; \underline{kg}$

Phys 371   Assgn # 7

2. Given $\epsilon = pc$ & $p = \dfrac{\pi \hbar}{L} n$

with $n = (n_x^2 + n_y^2 + n_z^2)^{1/2}$

We can write $\epsilon_F = p_F c = \dfrac{\pi \hbar}{L} c \, n_F$

$$\epsilon_F = \dfrac{\pi \hbar c}{L} \left(\dfrac{3N}{\pi} \right)^{1/3} = \pi \hbar c \left(\dfrac{3N}{\pi L^3} \right)^{1/3}$$

$$= \pi \hbar c \left(\dfrac{3n}{\pi} \right)^{1/3}$$

Where $n = \dfrac{N}{V} = N/L^3$ is particle no. density.

(b) We have $p = \dfrac{\pi \hbar}{L} n$

and $\epsilon(n) = p(n) c = \dfrac{\pi \hbar}{L} n c$

Total g.s. energy: $U_0 = 2 \sum_{n \le n_F} \epsilon(n)$ (Eq. 8)

$$U_0 = 2 \times \dfrac{1}{8} \int_0^{n_F} \epsilon_n \cdot 4\pi n^2 \, dn = \pi \int_0^{n_F} \left(\dfrac{\pi \hbar c}{L} \right) n^3 \, dn$$

$$= \pi \cdot \left(\dfrac{\pi \hbar c}{L} \right) \dfrac{n_F^4}{4}$$

But $n_F = \left(\dfrac{3N}{\pi} \right)^{1/3}$ and $\epsilon_F = \dfrac{\pi \hbar c}{L} n_F$

$\therefore U_0 = \pi \cdot \epsilon_F \cdot \left(\dfrac{3N}{\pi} \right) \cdot \dfrac{1}{4} = \dfrac{3}{4} N \epsilon_F$